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Conjunctions, Disjunctions, and Bell-Type Inequalities in Orthoalgebras

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A logicoalgebraic approach to the hidden variables issue is outlined. The motivation for studying Bell-type inequalities on orthoalgebras is given and the problem of conjunctions and disjunctions of compatible elements is discussed. A theorem concerning a very general form of Bell-type inequalities on orthoalgebras is proved.

1. INTRODUCTION

The objective of the logicoalgebraic approach to the foundations of quantum mechanics (see, e.g., Mackey, 1963; Jauch, 1968; Beltrametti and Cassinelli, 1981; Pták and Pulmannová, 1991) is to study the most basic mathematical structures underlying quantum theory and their relations to the corresponding structures of classical statistical mechanics. In the logicoalgebraic approach one deals mostly with two-valued observables, usually called propositions. Therefore, this approach seems to be particularly well suited to study Bell-type inequalities, in which such observables play an essential role. For many years the set of all propositions for a given physical system, i.e., a *logic* of a system, was usually assumed to be an *orthomodular lattice* (OML) or *orthomodular partially ordered set* (*orthomodular poset,* OMP) (for a detailed definition of OML, OMP, and for other relevant definitions the interested reader is referred to the above-mentioned texts), but nowadays also more general structures are studied. These mathematical structures possess the main features characteristic of the set $\mathcal{P}(\mathcal{H})$ of all projectors on a Hilbert space $\mathcal{H}$ and all basic notions of the Hilbert space quantum mechanics can be expressed within them. For example, if projectors P_A, P_B are repre-

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sented, respectively, by propositions a and b, then their orthogonality is equivalent to $a \leq b'$ (b' is the *orthocomplement* of b), and their commutativity is equivalent to the so-called compatibility of propositions a and b, denoted aCb, defined with the aid of *Mackey decomposition* (Mackey, 1963) as follows: aCb iff there exist pairwise orthogonal propositions $\tilde{a}$, $\tilde{b}$, and c such that $a = \tilde{a} \vee c$ and $b = \tilde{b} \vee c$, where "$\vee$" denotes the least upper bound (join) with respect to the given partial order.

States of a physical system are represented within the logicoalgebraic approach by *probability measures* defined on the studied ordered structure L, i.e., by functions p: $L \to [0, 1]$ that are additive on sequences of pairwise orthogonal propositions. These probability measures are usually themselves called *states* on L. The number $p(a)$ is usually interpreted as a probability of obtaining the desired outcome (one of the two possible outcomes) in an experiment designed to test the proposition a when the physical system is in the state represented by the measure p. *Pure* (*mixed*) *states* of a physical system are represented by probability measures which are not (are) convex combinations of other measures. A state p is called *dispersion-free on a proposition a* if either $p(a) = 1$ or $p(a) = 0$.

The very essence of any hidden-variables (HV) theory is to deprive quantum mechanics of its intrinsic statistical character. This consists in assuming that pure HV states should be dispersion-free on all propositions while quantum mechanical states, even pure ones, should be mixtures of HV states. The overwhelming success of quantum mechanics forces any HV advocate to require that a hypothetical HV theory should give results compatible with quantum mechanics. These two assumptions form the basis for numerous impossibility theorems produced to show that such a program is impossible (see, e.g., Belinfante, 1973; von Neumann, 1932; Jauch and Piron, 1963; Beltrametti and Cassinelli, 1981). However, up to now, there is no impossibility theorem without additional assumptions. Every "impossibility" theorem does not make impossible the very idea of a HV theory, but it excludes only a definite class of HV theories: those HV theories which fulfil the assumptions of a theorem. Therefore, any "impossibility" theorem has also positive content: It indicates what a *possible* HV theory could look like.

In many impossibility theorems additional assumptions were introduced in order to make the very structure of a hypothetical HV theory identical or very close to the structure of classical mechanics. We can mention here the linearity of observables assumed by von Neumann (1932), the Boolean structure of a HV logic assumed by Beltrametti and Cassinelli (1981) and by Santos (1986), or the requirement that states should fulfil the so-called Jauch–Piron condition assumed by Jauch and Piron (1963) and by Pykacz (1989). Because of these additional assumptions all these impossibility theorems exclude of course only those HV theories whose structures are very similar to the structure of classical mechanics. Since in general there is no reason to expect such similarity, gradually it has become obvious that more general cases should be also studied.

2. MOTIVATION FOR PASSING TO ORTHOALGEBRAS

A very general theorem yielding Bell-type inequalities free from the additional assumptions mentioned in the Introduction was proved by Pykacz and Santos (1991). In this theorem the very useful notion of *separation between propositions a and b in the state p*: $S_p(a, b) = p(a) + p(b) - 2p(a \wedge b)$, where $a \wedge b$ denotes the greatest lower bound (meet) of a and b, introduced in slightly less general form by Santos (1986), was utilized. Santos (1986) showed that if a logic L is a Boolean algebra, then for any $a, b \in L$ and any state p on L the triangle inequality

$$S_p(a, b) + S_p(b, c) \geq S_p(a, c) \tag{1}$$

holds and that this inequality is equivalent to the quadrilateral inequality

$$S_p(a_1, a_2) + S_p(b_1, a_2) + S_p(b_1, b_2) \geq S_p(a_1, b_2) \tag{2}$$

This quadrilateral inequality, when written in terms of single $p(a)$ and coincidence $p(a \wedge b)$ probabilities, takes the familiar form of the Clauser and Horne version of Bell's inequalities:

$$p(a_1 \wedge a_2) + p(b_1 \wedge a_2) + p(b_1 \wedge b_2) - p(a_1 \wedge b_2) \leq p(b_1) + p(a_2) \tag{3}$$

Therefore, inequalities of the type (1) and (2) were called by Pykacz and Santos (1991) *generalized Bell-type inequalities.*

The main result of Pykacz and Santos (1991) is the following:

Theorem 1. Let L be an orthomodular poset and let $a_1Ca_2Ca_3C \dots Ca_nCa_1$, i.e., $a_1, a_2, \dots, a_n$ are "circularly compatible" propositions. If p is a state which is dispersion-free on a pair (a_i, a_{i+1}), then the following generalized Bell-type inequality holds:

$$\sum_{\substack{k=1,\dots,n \\ k \neq i}} S_p(a_k, a_{k+1}) \geq S_p(a_i, a_{i+1}) \tag{4}$$

where we put $a_{n+1} = a_1$.

When $n = 4$, one obtains from (4) the quadrilateral inequality (2) and therefore the Clauser and Horne inequality (3).

Since all HV states are assumed to be dispersion-free on all propositions and the inequality (4) is preserved when mixtures of states are formed, one obtains immediately the following very general impossibility theorem (Pykacz and Santos, 1991):

Theorem 2. A hidden variables theory in which the set of propositions is an orthomodular poset and which would be able to give results compatible with quantum mechanics is impossible.

The "positive" content of Theorem 2 is the following: The structure of the set of propositions of a *possible* HV theory compatible with quantum mechanics should *not* be an OMP. This is a severe restriction, since a set of propositions in classical mechanics—a Boolean algebra—*is* an OMP. Therefore, for those who assert that "classical = Boolean," there is no way back from quantum mechanics to classical mechanics: The structure of a set of propositions of a possible HV theory will not be more "classical" than the structure of the set of quantum propositions. On the contrary, it should be even weaker than $\mathcal{P}(\mathcal{H})$—the well-known model of the logic of propositions about quantum systems consisting of projectors onto closed subspaces of a Hilbert space $\mathcal{H}$ which describes the system.

These results strongly indicate the necessity of studying Bell-type inequalities on more general algebraic structures encountered in the foundations of quantum mechanics, which, to the best of my knowledge, has not yet been done. The other motivation for such a program follows from the fact that in any EPR-type experiment performed to check Bell-type inequalities one deals with pairs of particles described with the aid of a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ associated, respectively, with each of the particles. However, contrary to the situation in the orthodox Hilbert space quantum mechanics, where the tensor product is a well-established notion, there is still no generally accepted notion of a tensor product in the logicoalgebraic approach (see, e.g., Aerts, 1984; Pulmannová, 1985; Foulis, 1989; Foulis and Bennett, 1993; Hudson and Pulmannová, 1993, 1994). Moreover, some of the proposed definitions of a tensor products of orthomodular posets were so restrictive that it was later shown that the objects which they define exist either only in noninteresting cases or even that they do not exist at all (Randall and Foulis, 1979). It was noticed by Foulis (1989) that the way out is either to modify the definition of a tensor product and define it only on a suitably chosen subcategory of orthomodular posets or to pass to the more general category of unital orthoalgebras.

The first possibility was chosen by Pykacz and Santos (1995), where Clauser–Horne-type inequalities were studied on tensor products of orthomodular lattices defined in the way proposed by Hudson and Pulmannová (1993).

The objective of the present paper is to begin the study of the second possibility, i.e., to study Bell-type inequalities on orthoalgebras. This should prepare the ground for studying Bell-type inequalities on tensor products of orthoalgebras and on still more general effect algebras, which, hopefully, will be accomplished in the future.

3. ORTHOALGEBRAS

Orthoalgebras were introduced by Randall and Foulis (1979) (see also Hardegree and Frazer, 1981) and recently they have attracted a lot of attention (see, e.g., Foulis *et al.*, 1992; Greechie, 1992; Foulis and Bennett, 1993; Dalla Chiara and Giuntini, 1994; Hamhalter *et al.*, 1995) as natural generalizations of orthomodular posets. A big part of this interest is undoubtedly connected with previously mentioned difficulties in forming tensor products of OMPs, which in the case of orthoalgebras can be more easily overcome. Definitions and results of the present section are quoted mainly from the papers by Foulis *et al.* (1992) and Foulis and Bennett (1993). These papers, as well as the paper by Hamhalter *et al.* (1995), provide also a lot of examples of interesting orthoalgebras and discuss in a detailed way their relations to OMPs, OMLs, and Boolean algebras.

Definition 1. An *orthoalgebra* is a system $(L, O, I, \oplus)$ consisting of a set L with two distinguished elements O and I and equipped with a partially defined binary operation $\oplus$, which we call an *orthogonal sum*, that satisfies the following conditions:

(a) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(b) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(c) For every $a \in L$, there exists a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = I$.

(d) If $a \oplus a$ is defined, then $a = O$.

Definition 2. Let L be an orthoalgebra and let $a, b \in L$.

(a) We say that a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined.

(b) If there is an element $c \in L$ such that $a \oplus c$ is defined and $a \oplus c = b$, we write $a \leq b$.

(c) For $a \in L$, the unique element $b \in L$ such that $a \oplus b = I$ is called the *orthocomplement* of a and is denoted a'.

It can be shown that $\leq$ is a partial order on L and that $O \leq a \leq I$ for all $a \in L$. In general the meet $a \wedge b$ and the join $a \vee b$ with respect to this partial order need not exist in L for arbitrary $a, b \in L$. However, $a \wedge a'$ and $a \vee a'$ always exist in L and they are equal to the least element O and the greatest element I, respectively. Moreover, $(a')' = a$, and if $a \leq b$, then $b' \leq a'$; therefore $': L \rightarrow L$ is an orthocomplementation in a bounded poset $(L, O, I, \leq)$. It can be shown that $a \perp b$ iff $a \leq b'$ and that if in this case $a \vee b$ exists in L, then it coincides with $a \oplus b$.

An orthomodular poset is an orthoalgebra L in which for any $a, b \in L$, if $a \oplus b$ is defined, then $a \oplus b = a \vee b$. An orthomodular lattice is an OMP in which $a \vee b$ exists for any $a, b \in L$. Finally, a Boolean algebra is an OML L such that $a \wedge b = 0 \Rightarrow a \perp b$ for all $a, b \in L$. This shows that orthoalgebras are on the next level of generality in the chain of structures which begins with Boolean algebras and proceeds via orthomodular lattices and orthomodular posets toward more general structures (effect algebras, Abelian RI-posets, . . .).

Definition 3. A *suborthoalgebra* of an orthoalgebra L is a subset of L which is closed under the orthogonal sum and orthocomplementation map inherited from L. If it is a Boolean algebra, it is called a *Boolean subalgebra* of L.

Definition 4. A subset C of an orthoalgebra L is said to be *compatible* and its elements are called *jointly compatible* iff there is a Boolean subalgebra B of L with $C \subseteq B$. We say that the elements $a, b \in L$ are *compatible* and denote it aCb if $\{a, b\}$ is a compatible subset of L. A subset D of L is said to be *orthogonal*, and the elements of D are called *jointly orthogonal*, iff D is compatible and its elements are pairwise orthogonal.

In orthomodular posets joint compatibility is defined as above, while pairwise compatibility is usually defined with the aid of Macky decomposition in the way mentioned in the Introduction. However, it can be proved that two elements of an OMP L are compatible iff they generate a Boolean subalgebra of L, so Definition 4 is equivalent to the usual definition of compatible elements when OA is an OMP.

Definition 5. A *state* on an orthoalgebra L is a function $p: L \rightarrow [0, 1]$ such that $p(I) = 1$ and $p(a \oplus b) = p(a) + p(b)$ iff $a \oplus b$ is defined. A state p on L is *dispersion-free* on $a \in L$ iff either $p(a) = 0$ or $p(a) = 1$.

4. CONJUNCTIONS AND DISJUNCTIONS IN ORTHOALGEBRAS

The problem of choosing a proper algebraic representation for logical operations of conjunction and disjunction is as old as the logicoalgebraic approach itself. In particular, this problem should be carefully considered when one wants to study logicoalgebraic versions of Bell-type inequalities since they always contain conjunctions of propositions; e.g., 'spin of the particle 1 is "up" *and* spin of the particle 2 is "down".'

In Boolean algebras, which are Lindenbaum–Tarski algebras of theories governed by rules of classical logic, conjunctions and disjunctions of propositions are represented, respectively, by meets and joins. Therefore, it seemed natural to keep this representation also in more general structures: orthomodular lattices, orthomodular posets, orthoalgebras, effect algebras, etc. However, even Birkhoff and von Neumann, the founding fathers of the logicoalgebraic approach, were not completely satisfied with this choice, since the above-mentioned representation becomes doubtful when one considers noncompatible propositions. In order to solve this problem some theorists (cf. Jammer, 1974, pp. 354–355) proposed to represent by meets and joins conjunctions and disjunctions of compatible propositions only. This works in orthomodular posets, where meets and joins of compatible propositions always exist, but not in orthoalgebras, where this property does not hold. In order to model conjunctions and disjunctions of pairwisely compatible elements of orthoalgebras Foulis (1994) proposed to utilize Mackey decomposition, which in OAs takes the following form:

Definition 6. A pair of elements a, b of an OA L is said to have a *Mackey decomposition* $\{\tilde{a}, \tilde{b}, c\}$ iff $\tilde{a}$, $\tilde{b}$, c are jointly orthogonal, $a = \tilde{a} \oplus c$ and $b = \tilde{b} \oplus c$.

It should be noticed (Foulis *et al.*, 1992; Foulis, 1994) that two elements of an OA are compatible iff they admit a Mackey decomposition, which, however, in general need not be unique.

The proposal of Foulis (1994) consists in treating the above-defined element c as representing the conjunction, and the element $\tilde{a} \oplus \tilde{b} \oplus c$ as representing the disjunction of compatible elements a and b if the pair a, b has the unique Mackey decomposition and if c is the meet of a and b in L. In what follows I shall argue that neither of these additional assumptions seems to be necessary (I am grateful to Prof. D. Foulis for providing me with some of the arguments).

Let us first consider the second assumption, i.e., the assumption that c is the meet of compatible elements a and b in L. It can be easily checked that in general c is a maximal lower bound of a and b in L, although it need not be the meet of a and b. Similarly, $\tilde{a} \oplus \tilde{b} \oplus c$ is a minimal upper bound of a and b in L, although it need not be the join of a and b. Therefore, if $a \wedge b$ and $a \vee b$ exist in L, then $a \wedge b = c$, $a \vee b = \tilde{a} \oplus \tilde{b} \oplus c$ (as happens in orthomodular posets), and the Mackey decomposition of a and b is unique. On the other hand, although the uniqueness of the Mackey decomposition of a and b does not imply that $a \wedge b$ and $a \vee b$ exist in L [cf., for example, the Wright triangle described by Foulis *et al.* (1992), Example 2.13, or by

Hamhalter *et al.* (1995), Example 1.6], the unique elements c and $\tilde{a} \oplus \tilde{b} \oplus c$ are in this case naturally distinguished among all other possible maximal lower bounds and minimal upper bounds of a and b. Therefore, since they coincide with $a \wedge b$ and $a \vee b$ if the latter exist, e.g., in Boolean algebras, OMLs, and OMPs, they are natural candidates for elements which could represent, respectively, the conjunction and the disjunction of a and b.

Let $\{\tilde{a}, \tilde{b}, c\}$ be the unique Mackey decomposition of compatible elements $a, b \in L$. Since c and $\tilde{a} \oplus \tilde{b} \oplus c$ are, respectively, a maximal lower bound and a minimal upper bound of a and b in L and since they obviously belong to any Boolean subalgebra B of L that contains a and b, they coincide, respectively, with the meet $a \wedge_B b$ and join $a \vee_B b$ of a and b in B. Every Boolean subalgebra B of L can be thought of as representing propositions which can be tested in a single experiment. Therefore, $c = a \wedge_B b$ and $\tilde{a} \oplus \tilde{b} \oplus c = a \vee_B b$ can be tested in each experiment which tests a and b, and they regain in this way their traditional meaning of the conjunction and disjunction of a and b in every "local" logic associated with a single experiment. It can be also mentioned that Younce (1990) proved that an orthoalgebra L has the *unique Mackey decomposition* (UMC) *property*, i.e., every compatible pair of elements of L has the unique Mackey decomposition, iff the intersection of every pair of Boolean subalgebras of L is again a Boolean subalgebra of L. This means that there exists a unique "minimal" experiment (i.e., the experiment involving minimal number of propositions) testing simultaneously compatible propositions a and b, their conjunction c, and their disjunction $\tilde{a} \oplus \tilde{b} \oplus c$.

Finally, let us assume for a while that for a given pair of compatible elements $a, b \in L$ their Mackey decomposition is not unique, i.e., that there exists a family of triples $\{\tilde{a}_i, \tilde{b}_i, c_i\}$ of different Mackey decompositions of a and b. Even in this case elements $c_i = a \wedge_i b$ and $\tilde{a}_i \oplus \tilde{b}_i \oplus c_i = a \vee_i b$, which now can be different in different Boolean subalgebras B_i containing a and b, can be interpreted as representing, respectively, conjunctions and disjunctions of a and b in different context-dependent measurements, i.e., in measurements of a and b whose results do not depend on a and b alone, but on the whole situation in which an experiment is performed. Such "contextual quantum logic" might prove indispensable in attempts to formalize logicoalgebraic structures of contextual hidden variables theories. However, I shall concentrate my attention in what follows on orthoalgebras with UMD property.

In conclusion I propose to modify Foulis' (1994) idea in the following way: *If $a, b \in L$ have the unique decomposition $\{\tilde{a}, \tilde{b}, c\}$, we refer to c as the conjunction of a and b in L and to $\tilde{a} \oplus \tilde{b} \oplus c$ as the disjunction of a and b in L, and denote them, respectively, $a\&b$ and $a|b$.*

5. BELL-TYPE INEQUALITIES ON ORTHOALGEBRAS

Now we are in position to generalize Theorem 1 to orthoalgebras. As a by-product we shall obtain a proof of that theorem that is much shorter than the proof given originally in Pykacz and Santos (1991) and we shall show that one of the original assumptions of Theorem 1 was in fact unnecessary.

Taking into account the argumentation of the previous section, let us define separation between compatible elements of an orthoalgebra in the following way:

Definition 7. If a, b are compatible elements of an orthoalgebra L and p is a state on L, then the *separation between a and b in the state p* is the number

$$S_p(a, b) = p(a) + p(b) - 2p(a\&b) \tag{5}$$

Let us note that this notion of separation coincides with Santos' (1986) original notion when L is an orthomodular poset.

Lemma 1. Let $a, b \in L$ have the unique Mackey decomposition $\{\tilde{a}, \tilde{b}, c\}$. Then

$$S_p(a, b) = p(\tilde{a}) + p(\tilde{b}) \tag{6}$$

Proof. Since any state p on L is additive on orthogonal elements, we have

$$\begin{aligned} S_p(a, b) &= p(\tilde{a} \oplus c) + p(\tilde{b} \oplus c) - 2p(c) \\ &= p(\tilde{a}) + p(c) + p(\tilde{b}) + p(c) - 2p(c) \\ &= p(\tilde{a}) + p(\tilde{b}) \end{aligned}$$

The generalization of Theorem 1 to orthoalgebras is as follows:

Theorem 3. Let L be an orthoalgebra with the UMD property and let $a_1Ca_2Ca_3C\ldots Ca_nCa_1$, i.e., $a_1, a_2, \ldots, a_n$ are "circularly compatible" elements of L. Then for any state p on L the following generalized Bell-type inequality holds:

$$\sum_{\substack{k=1,\ldots,n \\ k\neq l}} S_p(a_k, a_{k+1}) \geq S_p(a_i, a_{i+1}) \tag{7}$$

where we put $a_{n+1} = a_1$.

Proof. Due to Lemma 1,

$$\begin{aligned} &S_p(a_1, a_2) + S_p(a_2, a_3) + \cdots + S_p(a_{i-1}, a_i) + S_p(a_{i+1}, a_{i+2}) + \cdots + S_p(a_n, a_1) \\ &\quad = p(\tilde{a}_1) + 2p(\tilde{a}_2) + 2p(\tilde{a}_3) + \cdots + 2p(\tilde{a}_{i-1}) + p(\tilde{a}_i) + p(\tilde{a}_{i+1}) + 2p(\tilde{a}_{i+2}) \\ &\qquad + \cdots + 2p(\tilde{a}_n) + p(\tilde{a}_1) \\ &\quad \geq p(\tilde{a}_i) + p(\tilde{a}_{i+1}) = S_p(a_i, a_{i+1}) \end{aligned}$$

Since every orthomodular poset has the UMD property, this surprisingly short proof is of course valid also for orthomodular posets. Let us also note that the assumption made in Theorem 1 that a state p should be dispersion-free on at least ne pair of compatible propositions is unnecessary. Therefore, the consequences of Theorem 3 are stronger than those of Theorem 1 since conclusions are not conditioned on the assumption that hypothetical HV states should be dispersion-free on all propositions. In particular, the no-go Theorem 2 can be generalized as follows:

Theorem 4. A hidden variables theory in which the set of propositions is an orthoalgebra with the UMD property and which would be able to give results compatible with quantum mechanics is impossible.

Theorem 4 pushes the boundary of *possible* HV theories further away from Boolean algebras than was done by Theorem 2. However, let us note that in accordance with the possible "contextual" interpretation of orthoalgebras that do not have the UMD property, Theorem 4 does not make impossible contextual HV theories whose set of propositions has the structure of an orthoalgebra.

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